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Supplementary material - Bayesian quantile regression for a binary outcome model with correlated random effects

An application on crime recidivism in Canada

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1 Conditional densities for the non-blocked sampling in the BPQRCRE model

The binary panel quantile regression with correlated random effects (BPQRCRE) model, when stacked for individual i, can be written in the hierarchical form as follows,

$$
\begin{aligned}
& z_i = X_i\beta + \iota_{T_i}\alpha_i + w_i\theta + D_{\tau\sqrt{w_i}}u_i && \forall\ i=1,\cdots,n, \\
& y_{it} = \begin{cases} 1 & \text{if} z_{it} > 0, \\ 0 & \text{otherwise}, \end{cases} && \forall\ i=1,\cdots,n,;\ t=1,\cdots,T_i, \\
& \alpha_i \sim N\left(\overline{m}_i'\zeta, \sigma_\alpha^2\right), \qquad w_{it} \sim \mathscr{E}(1), && u_{it} \sim N(0,1), \\
& \beta \sim N_k(\beta_0, B_0), \qquad \sigma_\alpha^2 \sim IG\left(\frac{c_1}{2}, \frac{d_1}{2}\right), && \zeta \sim N_{k-1}(\zeta_0, C_0),
\end{aligned}
\tag{1}
$$

where the first three lines represent the model and distribution of the mixture variables, and the last line specifies the prior distribution of the model parameters. All the notations are as described in the paper. Using the Bayes's theorem, the complete posterior density is given by,

$$
\begin{aligned}
\pi(\beta,\alpha,z,w,\zeta,\sigma_\alpha^2 \mid y) \propto & \left\{ \prod_{i=1}^{n}\prod_{t=1}^{T_i} \Big[I(z_{it}>0)I(y_{it}=1) + I(z_{it}\le 0)I(y_{it}=0) \Big] \right\} \\
& \times \exp\left[-\frac{1}{2}\sum_{i=1}^{n} \left\{ (z_i - X_i\beta - \iota_{T_i}\alpha_i - w_i\theta)' D_{\tau\sqrt{w_i}}^{-2} (z_i - X_i\beta - \iota_{T_i}\alpha_i - w_i\theta) \right\} \right] \\
& \times \exp\left(-\sum_{i=1}^{n}\sum_{t=1}^{T_i} w_{it} \right) \left(2\pi\sigma_\alpha^2\right)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma_\alpha^2}\sum_{i=1}^{n} (\alpha_i - \overline{m}_i'\zeta)'(\alpha_i - \overline{m}_i'\zeta) \right] \\
& \times (2\pi)^{-\frac{k}{2}} |B_0|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\beta-\beta_0)' B_0^{-1} (\beta-\beta_0) \right] (2\pi)^{-\frac{k-1}{2}} |C_0|^{-\frac{1}{2}} \\
& \times \exp\left[-\frac{1}{2}(\zeta-\zeta_0)' C_0^{-1} (\zeta-\zeta_0) \right] \times \left(\sigma_\alpha^2\right)^{-\left(\frac{c_1}{2}+1\right)} \exp\left[-\frac{d_1}{2\sigma_\alpha^2} \right].
\end{aligned}
\tag{2}
$$

1.1 Conditional posterior density of β

The conditional posterior density $\pi(\beta \mid \alpha, z, w)$ can be derived from the complete posterior density given by equation (2) by collecting terms involving β as follows,

$$
\begin{aligned}
\pi(\beta \mid \alpha,z,w) \propto & \exp\left[-\frac{1}{2}\sum_{i=1}^{n} (z_i - X_i\beta - \iota_{T_i}\alpha_i - w_i\theta)' D_{\tau\sqrt{w_i}}^{-2} (z_i - X_i\beta - \iota_{T_i}\alpha_i - w_i\theta) \right] \\
& \times \exp\left[-\frac{1}{2}(\beta-\beta_0)' B_0^{-1} (\beta-\beta_0) \right].
\end{aligned}
$$

Let $G_i = (z_i - \iota_{T_i}\alpha_i - w_i\theta)$, and consider the expressions in the exponential without the $-1/2$ term. Then the quadratic expressions can be opened and regrouped as,

$$
\begin{aligned}
& \sum_{i=1}^{n} (G_i - X_i\beta)' D_{\tau\sqrt{w_i}}^{-2} (G_i - X_i\beta) + (\beta-\beta_0)' B_0^{-1} (\beta-\beta_0) \\
&= \sum_{i=1}^{n} G_i' D_{\tau\sqrt{w_i}}^{-2} G_i - 2\sum_{i=1}^{n} \beta' X_i' D_{\tau\sqrt{w_i}}^{-2} G_i + \sum_{i=1}^{n} \beta' X_i' D_{\tau\sqrt{w_i}}^{-2} X_i\beta + \beta' B_0^{-1}\beta - 2\beta' B_0^{-1}\beta_0 + \beta_0' B_0^{-1}\beta_0 \\
&= \beta' \left(\sum_{i=1}^{n} X_i' D_{\tau\sqrt{w_i}}^{-2} X_i + B_0^{-1} \right) \beta - 2\beta' \left(\sum_{i=1}^{n} X_i' D_{\tau\sqrt{w_i}}^{-2} G_i + B_0^{-1}\beta_0 \right) + \sum_{i=1}^{n} G_i' D_{\tau\sqrt{w_i}}^{-2} G_i + \beta_0' B_0^{-1}\beta_0.
\end{aligned}
$$

Our interest lies in the distribution of β, so all terms that do not involve β are absorbed into the proportionality constant. Applying the idea of completing the square to the previous equation we have,

$$\pi(\beta \mid \alpha, z, w) \propto \exp\left[-\frac{1}{2}\left(\beta-\widetilde{\beta}\right)'\widetilde{B}^{-1}\left(\beta-\widetilde{\beta}\right)\right], \quad \text{where,}$$
$$\widetilde{B}^{-1} = \left(\sum_{i=1}^{n} X_i' D_{\tau\sqrt{w_i}}^{-2} X_i + B_0^{-1}\right), \quad \text{and} \quad \widetilde{\beta} = \widetilde{B}\left(\sum_{i=1}^{n} X_i' D_{\tau\sqrt{w_i}}^{-2}(z_i - \iota_{T_i}\alpha_i - w_i\theta) + B_0^{-1}\beta_0\right),$$

which is recognized as the kernel of a normal distribution. Hence we can write,

$$\beta \mid \alpha, z, w \sim N_k\left(\widetilde{\beta}, \widetilde{B}\right),$$
$$\text{where } \widetilde{B}^{-1} = \left(\sum_{i=1}^{n} X_i' D_{\tau\sqrt{w_i}}^{-2} X_i + B_0^{-1}\right), \text{ and } \widetilde{\beta} = \widetilde{B}\left(\sum_{i=1}^{n} X_i' D_{\tau\sqrt{w_i}}^{-2}(z_i - \iota_{T_i}\alpha_i - w_i\theta) + B_0^{-1}\beta_0\right). \tag{3}$$

.

1.2 Conditional posterior density of α_i

Similar to β, the conditional posterior density $\pi\left(\alpha_i \mid \beta, z, w, \sigma_\alpha^2, \zeta\right)$ can also be derived from the complete posterior density, given by equation (2), by collecting terms involving α_i. This results in the following expression,

$$\pi\left(\alpha_i \mid \beta, z, w, \sigma_\alpha^2, \zeta\right) \propto \exp\left[-\frac{1}{2}(z_i - X_i\beta - \iota_{T_i}\alpha_i - w_i\theta)' D_{\tau\sqrt{w_i}}^{-2}(z_i - X_i\beta - \iota_{T_i}\alpha_i - w_i\theta)\right]$$
$$\times \exp\left[-\frac{1}{2\sigma_\alpha^2}\left(\alpha_i - \overline{m}_i'\zeta\right)'\left(\alpha_i - \overline{m}_i'\zeta\right)\right]$$

Let $H_i = (z_i - X_i\beta - w_i\theta)$ and consider the expressions in the exponential without the constant $-1/2$. Then the expression can be simplified as,

$$\begin{aligned}
&(H_i - \iota_{T_i}\alpha_i)' D_{\tau\sqrt{w_i}}^{-2}(H_i - \iota_{T_i}\alpha_i) + \sigma_\alpha^{-2}\left(\alpha_i - \overline{m}_i'\zeta\right)'\left(\alpha_i - \overline{m}_i'\zeta\right) \\
&= H_i' D_{\tau\sqrt{w_i}}^{-2} H_i - 2\alpha_i'\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2} H_i + \alpha_i'\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2}\iota_{T_i}\alpha_i + \sigma_\alpha^{-2}\alpha_i'\alpha_i - 2\sigma_\alpha^{-2}\alpha_i'\overline{m}_i'\zeta + \sigma_\alpha^{-2}\zeta'\overline{m}_i\overline{x}_i'\zeta \\
&= \alpha_i'\left(\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2}\iota_{T_i} + \sigma_\alpha^{-2}\right)\alpha_i - 2\alpha_i'\left(\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2}\iota_{T_i} + \sigma_\alpha^{-2}\overline{m}_i'\zeta\right) + H_i' D_{\tau\sqrt{w_i}}^{-2} H_i + \sigma_\alpha^{-2}\zeta'\overline{m}_i\overline{m}_i'\zeta.
\end{aligned}$$

We are interested in the distribution of α_i, so all terms that do not involve α_i are absorbed into the proportionality constant. Applying the idea of completing the square as in the previous equation we have,

$$\pi\left(\alpha_i \mid \beta, z, w, \sigma_\alpha^2, \zeta\right) \propto \exp\left[-\frac{1}{2}(\alpha_i - \widetilde{a})'\widetilde{A}^{-1}(\alpha_i - \widetilde{a})\right], \quad \text{where,}$$
$$\widetilde{A}^{-1} = \left(\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2}\iota_{T_i} + \sigma_\alpha^{-2}\right), \text{ and } \widetilde{a} = \widetilde{A}\left(\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2}(z_i - X_i\beta - w_i\theta) + \sigma_\alpha^{-2}\overline{m}_i'\zeta\right),$$

which is the kernel of a normal distribution. Hence, the conditional posterior density of α_i is given by,

$$\alpha_i \mid \beta, z, w, \sigma_\alpha^2, \zeta \sim N\left(\widetilde{a}, \widetilde{A}\right),$$
$$\text{where } \widetilde{A}^{-1} = \left(\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2}\iota_{T_i} + \sigma_\alpha^{-2}\right), \text{ and } \widetilde{a} = \widetilde{A}\left(\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2}(z_i - X_i\beta - w_i\theta) + \sigma_\alpha^{-2}\overline{m}_i'\zeta\right). \tag{4}$$

1.3 Conditional posterior density of w_{it}

The conditional posterior density $\pi\left(w_{it} \mid \beta, \alpha_i, z_{it}\right)$ can be obtained from the complete posterior density (equation 2) by collecting terms involving w_{it}. This is done as follows,

$$\begin{aligned}
\pi\left(w_{it} \mid \beta, \alpha_i, z_{it}\right) &\propto w_{it}^{-1/2} \exp\left[-\frac{1}{2}\left(z_{it}-x_{it}'\beta-\alpha_i-w_{it}\theta\right)'\frac{1}{\tau^2 w_{it}}\left(z_{it}-x_{it}'\beta-\alpha_i-w_{it}\theta\right)\right] \times \exp\left[-w_{it}\right] \\
&\propto w_{it}^{-1/2} \exp\left[-\frac{1}{2}\left\{2w_{it}+\frac{\left(z_{it}-x_{it}'\beta-\alpha_i-w_{it}\theta\right)^2}{\tau^2 w_{it}}\right\}\right] \\
&\propto w_{it}^{-1/2} \exp\left[-\frac{1}{2}\left\{\left(\frac{\theta^2}{\tau^2}+2\right)w_{it}+\frac{1}{\tau^2}\frac{\left(z_{it}-x_{it}'\beta-\alpha_i\right)^2}{w_{it}}\right\}\right],
\end{aligned}$$

where all terms not involving w_{it} have been absorbed in the proportionality constant. By comparing the last expression to that of a generalized inverse Gaussian distribution (GIG)[1], we have

$$\begin{aligned}
&w_{it} \mid \beta, \alpha_i, z_{it} \sim GIG\left(\frac{1}{2}, \widetilde{\lambda}_{it}, \widetilde{\eta}\right) \quad \text{for } i=1,\ldots,n, \text{ and } t=1,\ldots,T_i, \\
&\text{where } \widetilde{\lambda}_{it}=\left(\frac{z_{it}-x_{it}'\beta-\alpha_i}{\tau}\right)^2, \text{ and } \widetilde{\eta}=\left(\frac{\theta^2}{\tau^2}+2\right).
\end{aligned} \tag{5}$$

1.4 Conditional posterior density of σ_α^2

The conditional posterior density $\pi\left(\sigma_\alpha^2 \mid \alpha, \zeta\right)$ is also derived from the complete posterior density (equation 2) by collecting terms involving σ_α^2. This is done as follows,

$$\begin{aligned}
\pi\left(\sigma_\alpha^2 \mid \alpha, \zeta\right) &\propto \left(2\pi\sigma_\alpha^2\right)^{-\frac{n}{2}} \exp\left[-\frac{1}{2\sigma_\alpha^2}\sum_{i=1}^{n}\left(\alpha_i-\overline{m}_i'\zeta\right)'\left(\alpha_i-\overline{m}_i'\zeta\right)\right] \times \left(\sigma_\alpha^2\right)^{-\left(\frac{c_1}{2}+1\right)} \exp\left[-\frac{d_1}{2\sigma_\alpha^2}\right] \\
&\propto \left(2\pi\sigma_\alpha^2\right)^{-\frac{(n+c_1)}{2}+1} \exp\left[-\frac{1}{2\sigma_\alpha^2}\left\{d_1+\sum_{i=1}^{n}\left(\alpha_i-\overline{m}_i'\zeta\right)'\left(\alpha_i-\overline{m}_i'\zeta\right)\right\}\right],
\end{aligned}$$

which is the kernel of an inverse-gamma distribution (IG) with shape parameter $\widetilde{c}_1=n+c_1$ and scale parameter $\widetilde{d}_1=d_1+\sum_{i=1}^{n}\left(\alpha_i-\overline{m}_i'\zeta\right)'\left(\alpha_i-\overline{m}_i'\zeta\right)$. Therefore, the conditional posterior density can be written as,

$$\begin{aligned}
&\sigma_\alpha^2 \mid \alpha, \zeta \sim IG\left(\frac{\widetilde{c}_1}{2}, \frac{\widetilde{d}_1}{2}\right), \\
&\text{where } \widetilde{c}_1=\left(n+c_1\right), \text{ and } \widetilde{d}_1=d_1+\sum_{i=1}^{n}\left(\alpha_i-\overline{m}_i'\zeta\right)'\left(\alpha_i-\overline{m}_i'\zeta\right).
\end{aligned} \tag{6}$$

1.5 Conditional posterior density of ζ

The conditional posterior density $\pi\left(\zeta \mid \alpha, \sigma_\alpha^2\right)$ is derived from the complete posterior density (equation 2). Collecting terms involving ζ we obtain the following expression,

$$\pi\left(\zeta \mid \alpha, \sigma_\alpha^2\right) \propto \exp\left[-\frac{1}{2\sigma_\alpha^2}\sum_{i=1}^{n}\left(\alpha_i-\overline{m}_i'\zeta\right)'\left(\alpha_i-\overline{m}_i'\zeta\right)\right] \times \exp\left[-\frac{1}{2}\left(\zeta-\zeta_0\right)'C_0^{-1}\left(\zeta-\zeta_0\right).\right]$$

[1] The generalized inverse Gaussian distribution $GIG(p,a,b)$ is a three-parameters family of continuous probability distributions with probability density,

$$f(x)=\frac{(b/a)^{p/2}}{2K_p\left(\sqrt{ab}\right)}x^{(p-1)}\exp\left[-\frac{1}{2}\left(\frac{a}{x}+bx\right)\right], \qquad x>0,$$

where $K_p\left(.\right)$ is the modified Bessel function of the second kind, $a>0$, $b>0$ and p is a real parameter.

Focusing on the expressions in the exponential but without the $-1/2$ term, we have,

$$\begin{aligned}
&\sigma_\alpha^{-2}\sum_{i=1}^{n}\left(\alpha_i-\overline{m}_i'\zeta\right)'\left(\alpha_i-\overline{m}_i'\zeta\right)+(\zeta-\zeta_0)'C_0^{-1}(\zeta-\zeta_0)\\
&=\sigma_\alpha^{-2}\sum_{i=1}^{n}\alpha_i'\alpha_i-2\sigma_\alpha^{-2}\sum_{i=1}^{n}\zeta'\overline{m}_i\alpha_i+\sigma_\alpha^{-2}\sum_{i=1}^{n}\zeta'\overline{m}_i\overline{m}_i'\zeta+\zeta'C_0^{-1}\zeta-2\zeta'C_0^{-1}\zeta_0+\zeta_0'C_0^{-1}\zeta_0\\
&=\zeta'\left(\sigma_\alpha^{-2}\sum_{i=1}^{n}\overline{m}_i\overline{m}_i'+C_0^{-1}\right)\zeta-2\zeta'\left(\sigma_\alpha^{-2}\sum_{i=1}^{n}\overline{m}_i\alpha_i+C_0^{-1}\zeta_0\right)+\sigma_\alpha^{-2}\sum_{i=1}^{n}\alpha_i'\alpha_i+\zeta_0'C_0^{-1}\zeta_0
\end{aligned}$$

The interest lies in the distribution of ζ, so all terms that do not involve ζ are absorbed into the proportionality constant. Applying the idea of completing the square we have,

$$\begin{aligned}
&\pi\left(\zeta\mid\alpha,\sigma_\alpha^2\right)\propto\exp\left[-\frac{1}{2}\left(\zeta-\widetilde{\zeta}\right)'\widetilde{\Sigma}_\zeta^{-1}\left(\zeta-\widetilde{\zeta}\right)\right],\qquad\text{where}\\
&\widetilde{\Sigma}_\zeta^{-1}=\left(\sigma_\alpha^{-2}\sum_{i=1}^{n}\overline{m}_i\overline{m}_i'+C_0^{-1}\right),\quad\text{and}\quad\widetilde{\zeta}=\widetilde{\Sigma}_\zeta\left(\sigma_\alpha^{-2}\sum_{i=1}^{n}\overline{m}_i\alpha_i'+C_0^{-1}\zeta_0\right),
\end{aligned}$$

which is recognized as the kernel of a normal distribution. Hence, the conditional posterior density is,

$$\begin{gathered}
\zeta\mid\alpha,\sigma_\alpha^2\sim N_{k-1}\left(\widetilde{\zeta},\widetilde{\Sigma}_\zeta\right),\\
\text{where }\widetilde{\Sigma}_\zeta^{-1}=\left(\sigma_\alpha^{-2}\sum_{i=1}^{n}\overline{m}_i\overline{m}_i'+C_0^{-1}\right),\ \text{and }\widetilde{\zeta}=\widetilde{\Sigma}_\zeta\left(\sigma_\alpha^{-2}\sum_{i=1}^{n}\overline{m}_i\alpha_i'+C_0^{-1}\zeta_0\right).
\end{gathered}\tag{7}$$

1.6 Conditional posterior density of z_{it}

The conditional posterior density $\pi\left(z_{it}\mid\beta,\alpha,w,y_{it}\right)$ is obtained by collecting terms involving z_{it} from the complete posterior density (equation 2) and seen to have the following expression,

$$\begin{aligned}
\pi\left(z_{it}\mid\beta,\alpha,w,y_{it}\right)\propto{}&\left[I\left(z_{it}>0\right)I\left(y_{it}=1\right)+I\left(z_{it}\le 0\right)I\left(y_{it}=0\right)\right]\\
&\times\exp\left[-\frac{1}{2}\left(z_{it}-x_{it}'\beta-\alpha_i-w_{it}\theta\right)'\frac{1}{\tau^2w_{it}}\left(z_{it}-x_{it}'\beta-\alpha_i-w_{it}\theta\right)\right]\\
\propto{}&\left[I\left(z_{it}>0\right)I\left(y_{it}=1\right)+I\left(z_{it}\le 0\right)I\left(y_{it}=0\right)\right]\phi\left(z_{it}|x_{it}'\beta+\alpha_i+w_{it}\theta,\tau^2w_{it}\right),
\end{aligned}$$

where the notation $\phi\left(x|\mu,\Sigma\right)$ denotes the probability density function of a normal distribution with mean μ and variance Σ. Note that the observations are conditionally independent, so we can draw each z_{it} independently of each other. Consequently, we have,

$$\begin{aligned}
\pi\left(z_{it}\mid\beta,\alpha,w,y_{it}\right)&\propto I\left(z_{it}\le 0\right)\phi\left(z_{it}|x_{it}'\beta+\alpha_i+w_{it}\theta,\tau^2w_{it}\right)\quad\text{if }y_{it}=0,\\
\pi\left(z_{it}\mid\beta,\alpha,w,y_{it}\right)&\propto I\left(z_{it}>0\right)\phi\left(z_{it}|x_{it}'\beta+\alpha_i+w_{it}\theta,\tau^2w_{it}\right)\quad\text{if }y_{it}=1,
\end{aligned}$$

which implies that we can draw z_{it} as follows,

$$z_{it}\mid\beta,\alpha,w,y_{it}\sim\begin{cases}TN_{(-\infty,0]}\left(x_{it}'\beta+\alpha_i+w_{it}\theta,\tau^2w_{it}\right)&\text{if }y_{it}=0,\\ TN_{(0,\infty)}\left(x_{it}'\beta+\alpha_i+w_{it}\theta,\tau^2w_{it}\right)&\text{if }y_{it}=1,\end{cases}\tag{8}$$

where $TN_{[a,b]}\left(\mu,\sigma^2\right)$ denotes a normal distribution truncated to the interval $[a,b]$ with mean μ and variance σ^2.

2 Conditional densities for the blocked sampling in the BPQRCRE model

2.1 Joint posterior conditional density of (β, z_{it})

The Gibbs sampler presented in Section 1 is straightforward, however, there is a potential for poor mixing due to correlation between (β, α_i) and (z_i, α_i). This correlation arises because the variables corresponding to the parameters in α_i are often a subset of those in x'_{it}. Thus, by conditioning these items on one another, the mixing of the Markov chain will be slow.

To avoid this issue, we present an alternative algorithm which jointly samples (β, z_i) in one block within the Gibbs sampler. In particular, β is sampled marginally of α_i from a multivariate normal distribution. Thereafter, the latent variable z_i is sampled marginally of α_i from a truncated multivariate normal distribution.

2.1.1 Conditional posterior density of β

First, we derive the conditional posterior density of β, marginalized over the random effects α_i. Since $\alpha_i \sim N\left(\overline{m}'_i\zeta, \sigma^2_\alpha\right)$, we can write $\alpha_i = \overline{m}'_i\zeta + \xi_i$ with $\xi_i \sim N\left(0, \sigma^2_\alpha\right)$. Therefore, the model can be rewritten as follows,

$$
\begin{aligned}
z_i &= X_i\beta + \iota_{T_i}\alpha_i + w_i\theta + D_{\tau\sqrt{w_i}}u_i, \qquad \forall\, i = 1, ..., n, \\
&= X_i\beta + \iota_{T_i}\overline{m}'_i\zeta + w_i\theta + v_i,
\end{aligned}
$$

where $v_i = \iota_{T_i}\xi_i + D_{\tau\sqrt{w_i}}u_i$. Given the properties of the components that constitutes v_i, we have $E(v_i) = 0_{T_i}$. The covariance can be derived to have the following expression,

$$
\begin{aligned}
Cov(v_i) = E\left[v_i v'_i\right] &= E\left[\iota_{T_i}\xi_i\xi'_i\iota'_{T_i} + D_{\tau\sqrt{w_i}}u_iu'_iD_{\tau\sqrt{w_i}}\right] \\
&= \sigma^2_\alpha\iota_{T_i}\iota'_{T_i} + D^{-2}_{\tau\sqrt{w_i}} = \sigma^2_\alpha J_{T_i} + D^{-2}_{\tau\sqrt{w_i}} = \Omega_i.
\end{aligned}
$$

where $J_{T_i} = \iota_{T_i}\iota'_{T_i}$. The conditional posterior density $\pi\left(\beta \mid z, w, \sigma^2_\alpha, \zeta\right)$, marginally of α, has the expression,

$$
\begin{aligned}
\pi\left(\beta \mid z, w, \sigma^2_\alpha\right) \propto \exp&\left[-\frac{1}{2}\sum_{i=1}^{n}\left(z_i - X_i\beta - \iota_{T_i}\overline{m}'_i\zeta - w_i\theta\right)'\Omega_i^{-1}\left(z_i - X_i\beta - \iota_{T_i}\overline{m}'_i\zeta - w_i\theta\right)\right] \\
&\times \exp\left[-\frac{1}{2}(\beta - \beta_0)'B_0^{-1}(\beta - \beta_0)\right].
\end{aligned}
$$

Similar to Section 1.1, we open the quadratic expressions and collect the terms involving β to complete a square. This yields the following,

$$
\pi\left(\beta \mid z, w, \sigma^2_\alpha, \zeta\right) \propto \exp\left[-\tfrac{1}{2}\left(\beta - \widetilde{\beta}\right)'\widetilde{B}^{-1}\left(\beta - \widetilde{\beta}\right)\right]
$$

$$
\text{where } \widetilde{B}^{-1} = \left(\sum_{i=1}^{n} X'_i\Omega_i^{-1}X_i + B_0^{-1}\right), \text{ and } \widetilde{\beta} = \widetilde{B}\left(\sum_{i=1}^{n} X'_i\Omega_i^{-1}\left(z_i - \iota_{T_i}\overline{m}'_i\zeta - w_i\theta\right) + B_0^{-1}\beta_0\right),
$$

which is clearly the kernel of a normal distribution. Hence, the conditional posterior density is,

$$
\begin{aligned}
&\beta \mid z, w, \sigma^2_\alpha, \zeta \sim N_k\left(\widetilde{\beta}, \widetilde{B}\right) \\
&\text{where } \widetilde{B}^{-1} = \left(\sum_{i=1}^{n} X'_i\Omega_i^{-1}X_i + B_0^{-1}\right), \text{ and } \widetilde{\beta} = \widetilde{B}\left(\sum_{i=1}^{n} X'_i\Omega_i^{-1}\left(z_i - \iota_{T_i}\overline{m}'_i\zeta - w_i\theta\right) + B_0^{-1}\beta_0\right).
\end{aligned} \tag{9}
$$

2.1.2 Conditional posterior density of z_i

The conditional posterior density of the latent variable z, marginally of α, has the following expression,

$$\pi\left(z \mid \beta, z, w, \zeta, \sigma_\alpha^2, y\right) \propto \prod_{i=1}^{n} \left\{ \prod_{t=1}^{T_i} \left[I(z_{it} > 0) I(y_{it} = 1) + I(z_{it} \leq 0) I(y_{it} = 0) \right] \right\}$$
$$\times \exp\left[-\frac{1}{2} \left(z_i - X_i\beta - \iota_{T_i}\overline{m}_i'\zeta - w_i\theta \right)' \Omega_i^{-1} \left(z_i - X_i\beta - \iota_{T_i}\overline{m}_i'\zeta - w_i\theta \right) \right].$$

The above expression is recognized as the kernel of a truncated multivariate normal (TMVN) distribution. Hence, we can write,

$$z_i \mid \beta, w_i, \zeta, \sigma_\alpha^2, y_i \sim TMVN_{B_i}\left(X_i\beta + \iota_{T_i}\overline{m}_i'\zeta + w_i\theta, \Omega_i \right) \qquad \forall\ i = 1, \cdots, n,$$

where $B_i = (B_{i1} \times B_{i2} \times ... \times B_{iT_i})$ and B_{it} is the interval $(0, \infty)$ if $y_{it} = 1$ and the interval $(-\infty, 0]$ if $y_{it} = 0$. Sampling directly from a $TMVN$ is not possible, hence, as in Rahman and Vossmeyer (2019), we resort to the method proposed in Geweke (1991, 2005), which utilizes Gibbs sampling to make draws from a $TMVN$.

We apply the theorem on the conditional multivariate normal distribution (see Geweke (2005) p.171)[2] on full conditional $f(z_{it} \mid z_{i,-t}) = f\left(z_{it} \mid z_{i1}, ..., z_{i(t-1)}, z_{i(t+1)}, ..., z_{iT_i}\right)$ which we denote as $t \mid -t$. This results in:

$$\begin{pmatrix} z_{it} \\ z_{i,-t} \end{pmatrix} \sim N(\mu, \Sigma), \ \text{ where } \ \mu = \begin{pmatrix} x_{it}'\beta + \overline{m}_i'\zeta + w_{it}\theta \\ \left(X_i\beta + \iota_{T_i}\overline{m}_i'\zeta + w_i\theta\right)_{-t} \end{pmatrix}, \ \text{ and } \ \Sigma = \begin{bmatrix} \Sigma_{t,t} & \Sigma_{t,-t} \\ \Sigma_{-t,t} & \Sigma_{-t,-t} \end{bmatrix},$$

where $\left(X_i\beta + \iota_{T_i}\overline{m}_i'\zeta + w_i\theta\right)_{-t}$ is a column vector with t-th element removed. $\Sigma_{t,t}$ denotes the (t,t)-th element of Ω_i, $\Sigma_{t,-t}$ denotes the t-th row of Ω_i with element in the t-th column removed and $\Sigma_{-t,-t}$ is the Ω_i matrix with t-th row and t-th column removed. Then, the distribution of z_{it} conditional on $z_{i,-t}$ is normal with conditional mean and conditional variance equal to,

$$\mu_{t|-t} = x_{it}'\beta + \overline{m}_i'\zeta + w_{it}\theta + \Sigma_{t,-t}\Sigma_{-t,-t}^{-1}\left(z_{i,-t} - \left(X_i\beta + \iota_{T_i}\overline{m}_i'\zeta + w_i\theta\right)_{-t} \right),$$
$$\Sigma_{t|-t} = \Sigma_{t,t} - \Sigma_{t,-t}\Sigma_{-t,-t}^{-1}\Sigma_{-t,t}.$$

We can then construct a Markov chain which continuously draws from $f(z_{it} \mid z_{i,-t})$ subject to the bound $(0, \infty)$ if $y_{it} = 1$ or $(-\infty, 0]$ if $y_{it} = 0$. This is done by sampling z_i at the j-th pass of the MCMC iteration using a series of conditional posterior distributions as follows:

$$z_{it}^j \mid z_{i1}^j, ..., z_{i(t-1)}^j, z_{i(t+1)}^j, ..., z_{iT_i}^j \sim TN_{B_{it}}\left(\mu_{t|-t}, \Sigma_{t|-t}\right), \quad \text{for } t = 1, \cdots, T_i, \tag{10}$$

where TN denotes a truncated normal distribution. The terms $\mu_{t|-t}$ and $\Sigma_{t|-t}$ are the conditional mean and variance defined above and $z_{i,-t}$ at the $j-$th pass is $z_{i,-t}^j = \left(z_{i1}^j, ..., z_{i(t-1)}^j, z_{i(t+1)}^{j-1}, ..., z_{iT_i}^{j-1}\right)'$.

2.2 Other conditional posterior densities

The derivations for the conditional posterior densities of α_i, w_{it}, σ_α^2 and ζ remains unaltered as presented earlier from Section 1.2 to Section 1.5. For the sake of completion, we present the conditional

[2] Let $\begin{pmatrix} x \\ y \end{pmatrix} \sim N(\mu, \Sigma)$ with $\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$ and $\Sigma = \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}$, then the distribution of y conditional on x is normal $N\left(\mu_{y.x}, \Sigma_{y.x}\right)$ with $\mu_{y.x} = \mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}(x - \mu)$ and $\Sigma_{y.x} = \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}$.

distributions below once more.

$$\begin{gathered}\alpha_i \mid \beta, z, w, \sigma_\alpha^2, \zeta \sim N\left(\widetilde{a}, \widetilde{A}\right), \quad \text{for } i = 1, \cdots, n, \\ \text{where } \widetilde{A}^{-1} = \left(\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2} \iota_{T_i} + \sigma_\alpha^{-2}\right), \text{ and } \widetilde{a} = \widetilde{A}\left(\iota_{T_i}' D_{\tau\sqrt{w_i}}^{-2} (z_i - X_i\beta - w_i\theta) + \sigma_\alpha^{-2} \overline{m}_i' \zeta\right).\end{gathered} \tag{11}$$

$$\begin{gathered}w_{it} \mid \beta, \alpha_i, z_{it} \sim GIG\left(\tfrac{1}{2}, \widetilde{\lambda}_{it}, \widetilde{\eta}\right), \quad \text{for } i = 1, \cdots, n, \text{ and } t = 1, \cdots, T_i, \\ \text{where } \widetilde{\lambda}_{it} = \left(\tfrac{z_{it} - x_{it}'\beta - \alpha_i}{\tau}\right)^2, \text{ and } \widetilde{\eta} = \left(\tfrac{\theta^2}{\tau^2} + 2\right).\end{gathered} \tag{12}$$

$$\begin{gathered}\sigma_\alpha^2 \mid \alpha, \zeta \sim IG\left(\tfrac{\widetilde{c}_1}{2}, \tfrac{\widetilde{d}_1}{2}\right), \\ \text{where } \widetilde{c}_1 = (n + c_1), \text{ and } \widetilde{d}_1 = d_1 + \sum_{i=1}^{n} (\alpha_i - \overline{m}_i'\zeta)' (\alpha_i - \overline{m}_i'\zeta).\end{gathered} \tag{13}$$

$$\begin{gathered}\zeta \mid \alpha, \sigma_\alpha^2 \sim N_{k-1}\left(\widetilde{\zeta}, \widetilde{\Sigma}_\zeta\right), \\ \text{where } \widetilde{\Sigma}_\zeta^{-1} = \left(\sigma_\alpha^{-2} \sum_{i=1}^{n} \overline{m}_i \overline{m}_i' + C_0^{-1}\right), \text{ and} \widetilde{\zeta} = \widetilde{\Sigma}_\zeta \left(\sigma_\alpha^{-2} \sum_{i=1}^{n} \overline{m}_i \alpha_i' + C_0^{-1} \zeta_0\right).\end{gathered} \tag{14}$$

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